Tracce delle soluzioni

A1.

Impedenza del parallelo capacità e resistenza Z_p:

$$Z_p = \frac{R \cdot \frac{1}{sC}}{R + \frac{1}{sC}} = \frac{R}{1+RCs}$$

Impedenza di trasferimento del tripolo Z_t:

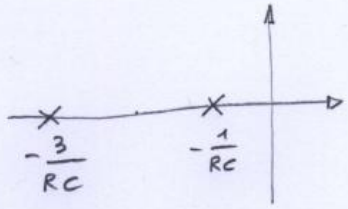
$$Z_t = R + R + \frac{R \cdot R}{Z_p} = 2R + \frac{R^2}{\frac{R}{1+RCs}} = 2R + R(1+RCs)$$

$$G(s) = -\frac{Z_p}{Z_t} = -\frac{\frac{R}{1+RCs}}{2R + R(1+RCs)}$$

①

$$G(s) = -\frac{1}{(1+RCs)(3+RCs)}$$

② Poli: $-\frac{1}{RC}$, $-\frac{3}{RC}$



$$\text{modi di } \Sigma = \left\{ e^{-\frac{t}{RC}}, e^{-\frac{3t}{RC}} \right\}$$

③

$$(1+RCs)(3+RCs) = 3 + RCs + 3RCs + (RC)^2 s^2 =$$
$$= (RC)^2 s^2 + 4RCs + 3$$

$$G(s) = \frac{-1}{(RC)^2 s^2 + 4RCs + 3}$$

eq. diff. $\quad (RC)^2 D^2 y + 4RC\, Dy + 3y = -u$

A2.

$$\begin{cases} m D^2 x_1 = f - k x_1 - b D x_1 + k(x_2 - x_1) + b(D x_2 - D x_1) \\ m D^2 x_2 = -k(x_2 - x_1) - b(D x_2 - D x_1) \end{cases}$$

$$\begin{cases} (bD + k) x_2 = m D^2 x_1 + 2bD x_1 + 2k x_1 - f \\ (m D^2 + bD + k) x_2 = bD x_1 + k x_1 \end{cases}$$

$$(m D^2 + bD + k)(m D^2 x_1 + 2bD x_1 + 2k x_1 - f) =$$
$$= (bD + k)(bD x_1 + k x_1)$$

$$(m D^2 + bD + k)(m D^2 + 2bD + 2k) x_1 - (m D^2 + bD + k) f =$$
$$= (bD + k)^2 x_1$$

$$(m^2 D^4 + 3bm D^3 + (3km + 2b^2) D^2 + 4bk D + 2k^2) x_1$$
$$- (b^2 D^2 + 2bk D + k^2) x_1 = (m D^2 + bD + k) f$$

(1)

$$m^2 D^4 x_1 + 3bm D^3 x_1 + (3km + b^2) D^2 x_1 + 2bk D x_1 + k^2 x_1 =$$
$$= m D^2 f + bD f + k f$$

(2)

$$G(s) = \frac{m s^2 + b s + k}{m^2 s^4 + 3bm s^3 + (3km + b^2) s^2 + 2bk s + k^2}$$

B1.

Vedi le dispense del corso.

B2.

1.

$u(t)=9,\ Du(t)=0,\ D^2u(t)=0$

$\Rightarrow D^2u(t)+Du(t)+u(t)=9 \quad \forall t<0.$

$y(t)=1+e^{-3t},\ Dy(t)=-3e^{-3t},\ D^2y(t)=9e^{-3t}$

$\Rightarrow D^2y(t)+6Dy(t)+9y(t)=9e^{-3t}+6\left(-3e^{-3t}\right)+9\left(1+e^{-3t}\right)=9 \quad \forall t<0.$

2.

Calcolo delle condizioni iniziali al tempo $0-$:

$u(0-)=9,\ Du(0-)=0; \quad y(0-)=1+1=2,\ Dy(0-)=-3$

Equazione diff. interpretata in senso distribuzionale:

$D^{*2}y+6D^*y+9y=D^{*2}u+D^*u+u$

Applichiamo la trasformata di Laplace:

$$s^2Y(s)-sy(0-)-Dy(0-)+6\left[sY(s)-y(0-)\right]+9Y(s)=$$
$$=s^2U(s)-su(0-)-Du(0-)+\left[sU(s)-u(0-)\right]+U(s)$$

$s^2Y-2s+3+6\left[sY-2\right]+9Y=s^2U-9s+sU-9+U$

$(s^2+6s+9)Y=(s^2+s+1)U-7s$

$$Y(s)=\frac{s^2+s+1}{s^2+6s+9}U(s)-\frac{7s}{s^2+6s+9}=\frac{s^2+s+1}{s^2+6s+9}\cdot\frac{18}{s}-\frac{7s}{s^2+6s+9}=$$
$$=\frac{18(s^2+s+1)-7s^2}{s(s^2+6s+9)}=\frac{11s^2+18s+18}{s(s^2+6s+9)}=\frac{k_1}{s}+\frac{k_{21}}{(s+3)^2}+\frac{k_{22}}{s+3}$$

$$k_1=\left.\frac{11s^2+18s+18}{(s^2+6s+9)}\right|_{s=0}=2 \qquad k_{21}=\left.\frac{11s^2+18s+18}{s}\right|_{s=-3}=-21$$

$k_1+k_{22}=11 \quad \Rightarrow \quad k_{22}=9$

$y(t)=2-21\cdot t\cdot e^{-3t}+9\cdot e^{-3t} \quad , \ t\geq 0$

B3.

modi di $\Sigma = \left\{e^{-2t}\sin(3t+\varphi_1), te^{-2t}\sin(3t+\varphi_2), e^{5t}, te^{5t}, t^2e^{5t}, t^3e^{5t}, e^{-10t}\right\}$

$$y_{\text{lib.}}(t)=c_1e^{-2t}\sin(3t+\varphi_1)+c_2te^{-2t}\sin(3t+\varphi_2)+c_3e^{5t}+c_4te^{5t}+c_5t^2e^{5t}+c_6t^3e^{5t}+c_7e^{-10t}$$
$$c_i\in\mathbb{R}, i=1,\dots,7;\ \varphi_1,\varphi_2\in\mathbb{R}.$$

C1.

1° metodo :

Calcolo di $y(t)$ per $0 \le t < 1$:

$$u(t) = 1,\quad U(s) = \frac{1}{s} \quad \Rightarrow \quad Y(s) = G(s)U(s) = \frac{4}{s(s+1)(s+2)}$$

$$Y(s) = \frac{k_1}{s} + \frac{k_2}{s+1} + \frac{k_3}{s+2};$$

$$k_1 = \left.\frac{4}{(s+1)(s+2)}\right|_{s=0} = 2\ ;\quad k_2 = \left.\frac{4}{s(s+2)}\right|_{s=-1} = -4\ ;\quad k_3 = \left.\frac{4}{s(s+1)}\right|_{s=-2} = 2\ ;$$

$$y(t) = 2 - 4e^{-t} + 2e^{-2t}$$

Calcolo di $y(t)$ per $t \ge 1$:

$$y(t) = c_1 e^{-t} + c_2 e^{-2t}$$

Considerato che $\rho = 2$ e $y \in \overline{C^{\rho-1,\infty}}(\mathbb{R}) \quad \Rightarrow \quad y \in \overline{C^{1,\infty}}(\mathbb{R})$

$$\Rightarrow \begin{cases} y(1-) = y(1+) \\ Dy(1-) = Dy(1+) \end{cases},$$

$Dy(t) = -c_1 e^{-t} - 2c_2 e^{-2t}$ per $t \ge 1$; $\qquad Dy(t) = 4e^{-t} - 4e^{-2t}$ per $0 \le t < 1$

$$\begin{cases} 2 - 4e^{-1} + 2e^{-2} = c_1 e^{-1} + c_2 e^{-2} \\ 4e^{-1} - 4e^{-2} = -c_1 e^{-1} - 2c_2 e^{-2} \end{cases} \quad \Rightarrow \quad c_1 = 4e - 4\ ;\ c_2 = 2 - 2e^2\ ;$$

$$y(t) = 4(e-1)\cdot e^{-t} + 2(1-e^2)\cdot e^{-2t}$$

2° metodo :

$$u(t) = 1(t) - 1(t-1), \quad U(s) = \frac{1}{s} - \frac{1}{s} e^{-s}$$

$$\Rightarrow \quad Y(s) = G(s)U(s) = \frac{4}{(s+1)(s+2)} \left[\frac{1}{s} - \frac{1}{s} e^{-s} \right]$$

$$Y(s) = \frac{4}{s(s+1)(s+2)} - \frac{4}{s(s+1)(s+2)} e^{-s}$$

$$y(t) = \mathcal{L}^{-1}\left[\frac{4}{s(s+1)(s+2)} \right] - \mathcal{L}^{-1}\left[\frac{4}{s(s+1)(s+2)} e^{-s} \right]$$

$$\mathcal{L}^{-1}\left[\frac{4}{s(s+1)(s+2)} \right] = \mathcal{L}^{-1}\left[\frac{2}{s} - \frac{4}{s+1} + \frac{2}{s+2} \right] = 2 - 4e^{-t} + 2e^{-2t} \quad \text{per } t \geq 0$$

Digressione: dal teorema di traslazione nel tempo

$$\mathcal{L}\left[f(t-t_0) \cdot 1(t-t_0) \right] = e^{-t_0 s} F(s) \; ; \quad F(s) := \mathcal{L}\left[f(t) \right]$$

$$\Rightarrow f(t-t_0) \cdot 1(t-t_0) = \mathcal{L}^{-1}\left[e^{-t_0 s} F(s) \right]$$

$$\mathcal{L}^{-1}\left[\frac{4}{s(s+1)(s+2)} e^{-s} \right] = \left[2 - 4e^{-(t-1)} + 2e^{-2(t-1)} \right] \cdot 1(t-1) \quad \text{per } t \geq 0$$

$$y(t) = 2 - 4e^{-t} + 2e^{-2t} - \left[2 - 4e^{-(t-1)} + 2e^{-2(t-1)} \right] \cdot 1(t-1)$$

da cui per $0 \leq t < 1$: $y(t) = 2 - 4e^{-t} + 2e^{-2t}$

e per $t \geq 1$: $y(t) = 2 - 4e^{-t} + 2e^{-2t} - \left[2 - 4e^{-(t-1)} + 2e^{-2(t-1)} \right] =$

$$= (-4 + 4e)e^{-t} + (2 - 2e^2)e^{-2t}$$

C2.

$$G_{ry}(s) = \frac{L(s)}{1 + L(s)} = \frac{\frac{12}{s(s+4)}}{1 + \frac{12}{s(s+4)}} = \frac{12}{s(s+4)+12} = \frac{12}{s^2 + 4s + 12}$$

Dal confronto $\dfrac{12}{s^2 + 4s + 12} = \dfrac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}$

$$\omega_n = \sqrt{12} \quad \Rightarrow \quad T_s \simeq \frac{1.8}{\omega_n} = 0,52 \;\; \text{sec.}$$

$$2\delta\omega_n = 4 \;\Rightarrow\; \delta\omega_n = 2 \;\Rightarrow\; T_a = \frac{3}{\delta\omega_n} = 1,5 \;\; \text{sec.}$$

$$\delta = \frac{2}{\sqrt{12}} = 0,5774 \quad \Rightarrow \quad S = 100 \exp\left(-\frac{\delta\pi}{\sqrt{1-\delta^2}} \right) = 10,8\%$$